Fuzzy Inference and Defuzzification

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Lecture Outline

- Crisp Rules Revision
- Fuzzy Sets revision
- Fuzzy Inference
- Fuzzy Rules
- Fuzzy Composition
- Defuzzification

Crisp Rules

- Consist of antecedents and consequents
- Each part of an antecedent is a logical expression
  - e.g. A > 0.5, light is on
- Consequent will be asserted if antecedent is true
  - IF (Presentation is Dull) AND (Voice is Monotone)
  - THEN Lecture is boring

Crisp Rules

- Only one rule at a time allowed to fire
- A rule will either fire or not fire
- Have problems with uncertainty
- Have problems with representing concepts like *small*, *large*, *thin*, *wide*
- Sequential firing of rules also a problem
  - order of firing

Fuzzy Sets

- Supersets of crisp sets
- Items can belong to varying degrees
  - degrees of membership
  - [0,1]
- Fuzzy sets defined two ways
  - membership functions
    - MF
  - sets of ordered pairs

Fuzzy Sets

- Membership functions (MF)
- Mathematical functions
- Return the degree of membership in a fuzzy set
- Many different types in existence
  - Gaussian
  - Triangular

Fuzzy Sets

- Can also be described as sets of ordered pairs
- Pair Crisp,Fuzzy values
  - A={(0,1.0),(1,1.0),(2,0.75),(3,0.5),(4,0.25),(5,0.0),(6,0.0),(7,0.0),(8,0.0),(9,0.0),(10,0.0)}
- With enough pairs, can approximate any MF

Fuzzy Sets

- Fuzzification
- Process of finding the degree of membership of a value in a fuzzy set
- Can be done by
  - MF
  - Interpolating set of pairs

Fuzzy Rules

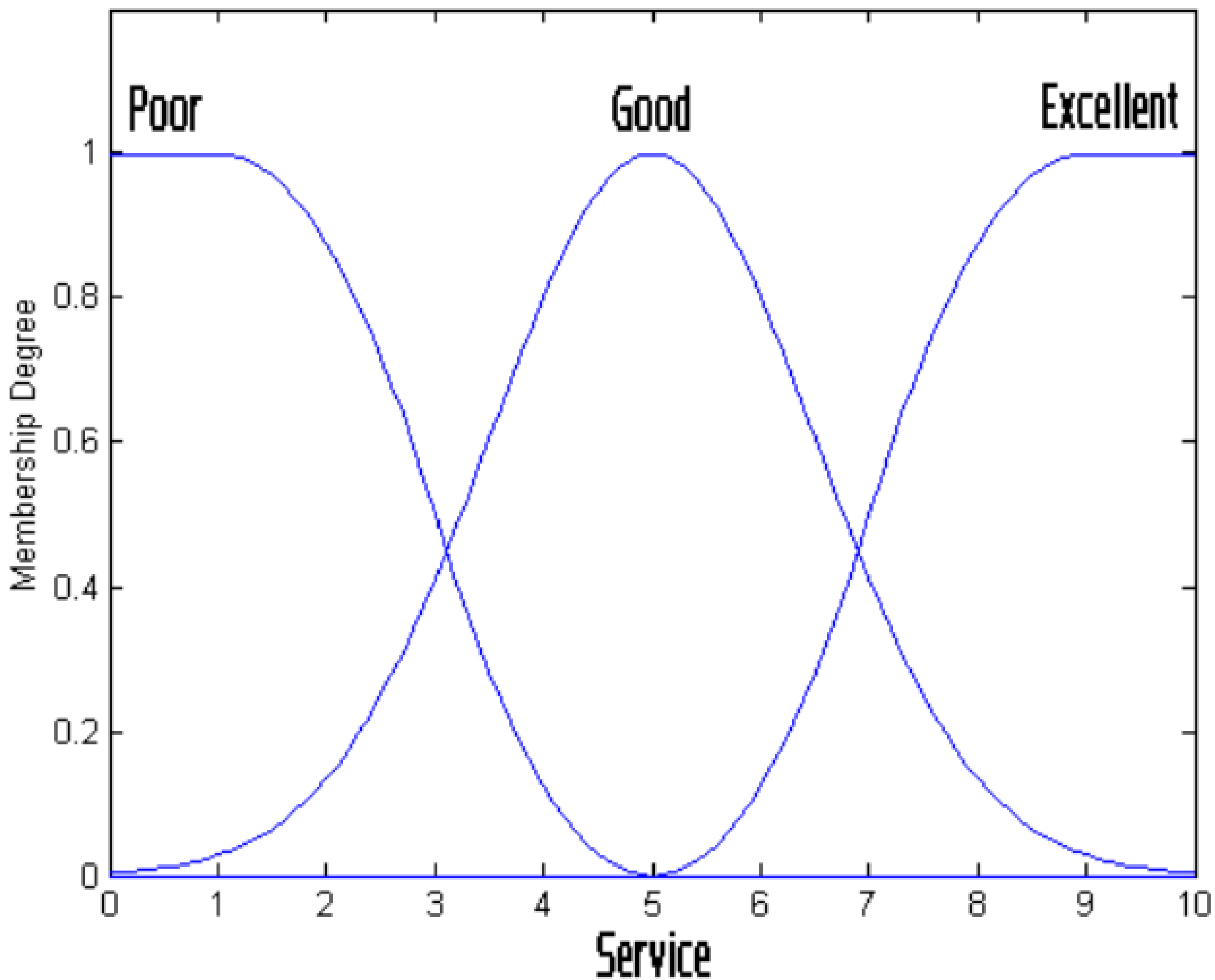
- Also have antecedents and consequents
- Both deal with partial truths
- Antecedents match fuzzy sets
- Consequents assign fuzzy sets
- Fuzzy rules can have weightings
  - [0,1]
  - importance of rule
  - commonly set to 1

Fuzzy Rules

- Restaurant tipping example
- Antecedent variables are
  - quality of service
  - quality of food
- Consequent variables are
  - Tip

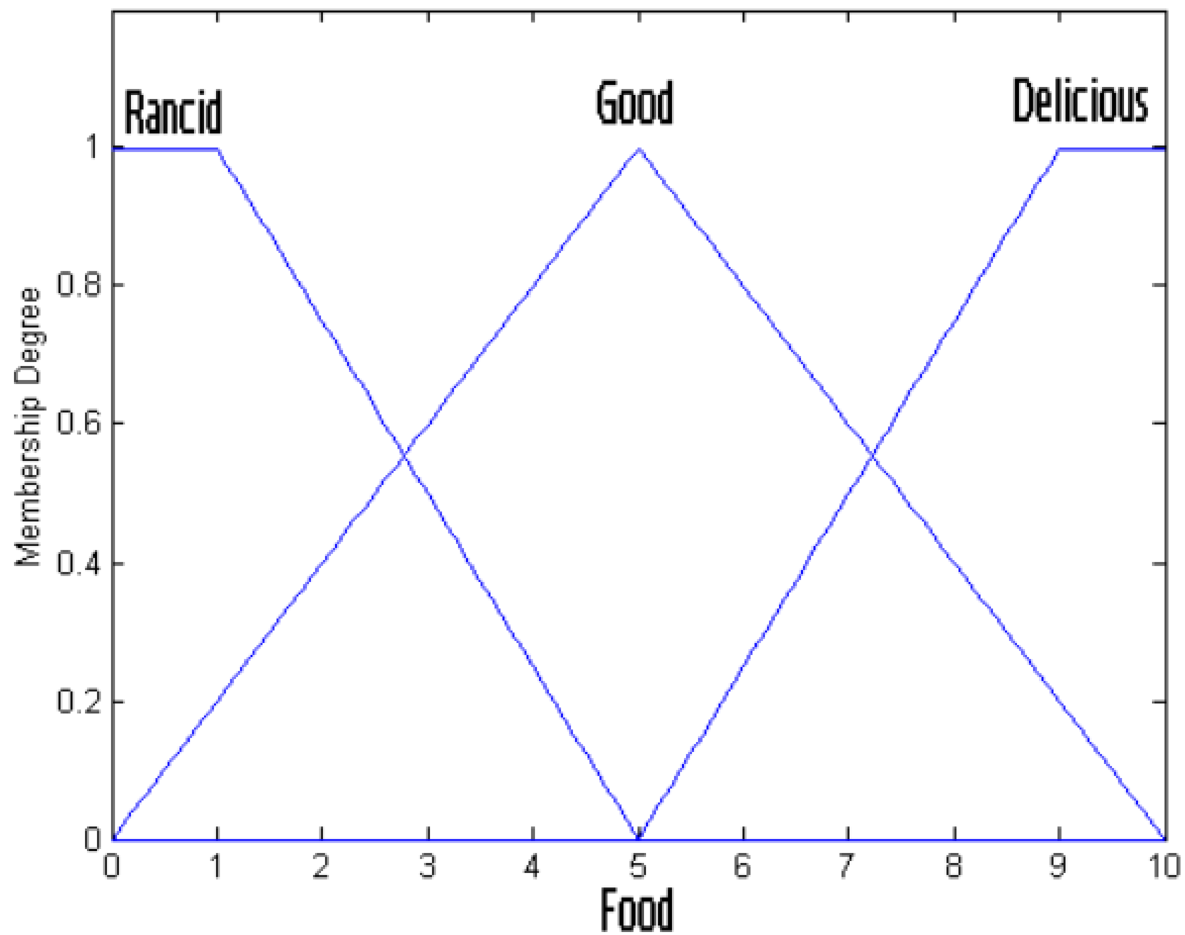
Fuzzy Rules

- Service can b
  - Poor
  - good
  - excellent
- Universe of discourse is 0 10



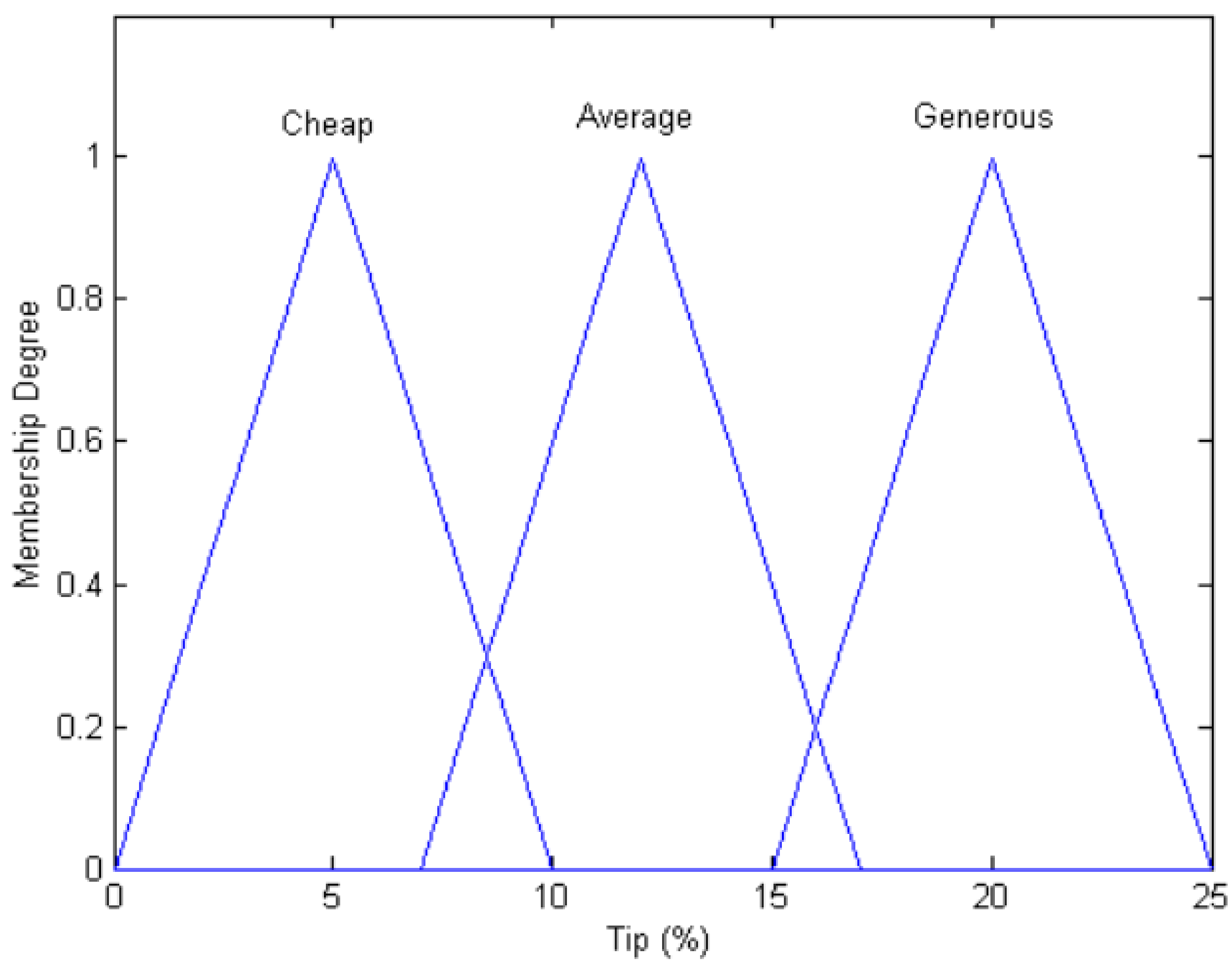
Fuzzy Rules

- Food can be
  - rancid
  - good
  - delicious
- Universe of discourse is 0-10



Fuzzy Rules

- Tip can be
  - cheap
  - average
  - generous
- Universe of discourse is 0-25
  - % tip



Fuzzy Rules

- Rules for the tipping system
  - IF service is poor or food is rancid
  - THEN tip is cheap
  - IF service is good
  - THEN tip is average
  - IF service is excellent or food is delicious
  - THEN tip is generous

Fuzzy Inference

- Infers fuzzy conclusions from fuzzy facts
- Matches facts against fuzzy antecedents
- Assigns fuzzy sets to outputs
- Three step process
  - fuzzify the inputs (fuzzification)
  - apply fuzzy logical operators across antecedents
  - apply implication method

Fuzzy Inference

- Implication is really two different processes
  - inference
  - composition
- Inference is the matching of facts to antecedents
- Results in the truth value of each rule
  - degree of support
  - Alpha

Fuzzy Inference

- Assigns fuzzy sets to each output variable
- Fuzzy sets assigned to different degrees
- Determined by degree of support for rule
- Methods for assigning (inferring) sets
  - min
  - Product

Fuzzy Inference

- Min inference
- Cut output MF at degree of support

$$\mu(v)' = min(z, \mu(v))$$

Where:

- μ is the output MF
- μ' is the inferred MF
- *v* is the value being fuzzified
- *z* is the degree of support

Fuzzy Inference

- Product inferencing
- Multiply output MF by degree of support

$$\mu(v)' = z\mu(v)$$

Tipping Example

- Assume
  - service is poor
    - score of 2
  - food is delicious
    - score of 8
- How do we perform fuzzy inference with these values?

Tipping Example

- Firstly, fuzzify the input values
- Service fuzzifies to
  - Poor 0.8
  - Good 0.2
  - Excellent 0.0
- Food fuzzifies to
  - Rancid 0.0
  - Good 0.4
  - Delicious 0.6

Tipping Example

- Now, calculate the degree of support for each rule
- Rule 1:
  - IF service is poor or food is rancid
  - poor = 0.8
  - rancid = 0.0
  - max(0.8, 0.0) = 0.8
  - Degree of support = 0.8

Tipping Example

- Rule 2
  - IF service is good
  - good = 0.2
  - max(0.2) = 0.2
  - Degree of support = 0.2

Tipping Example

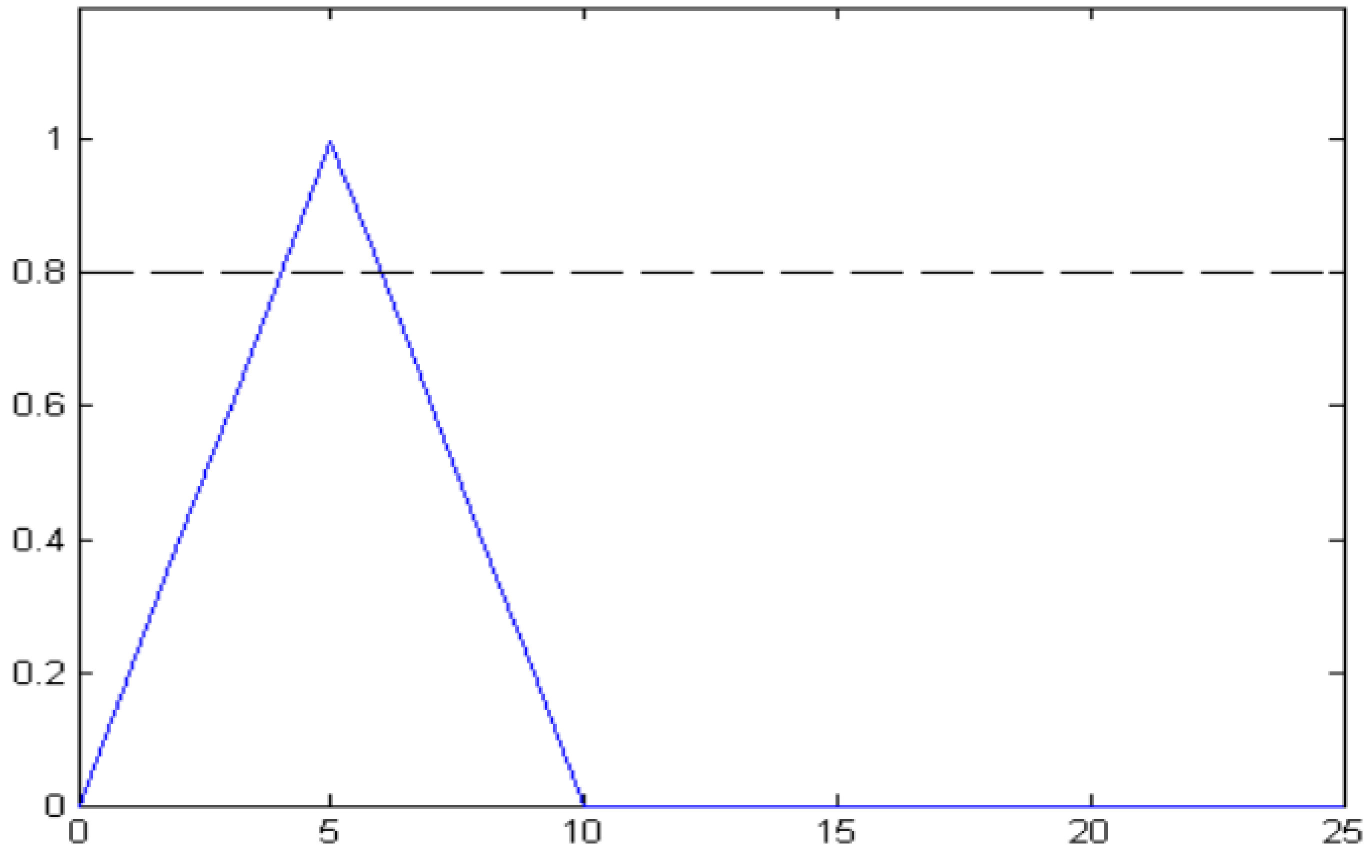
- Rule 3
  - IF service is excellent or food is delicious
  - excellent = 0.0
  - delicious = 0.6
  - max(0.0, 0.6) = 0.6
  - Degree of support = 0.6

Tipping Example

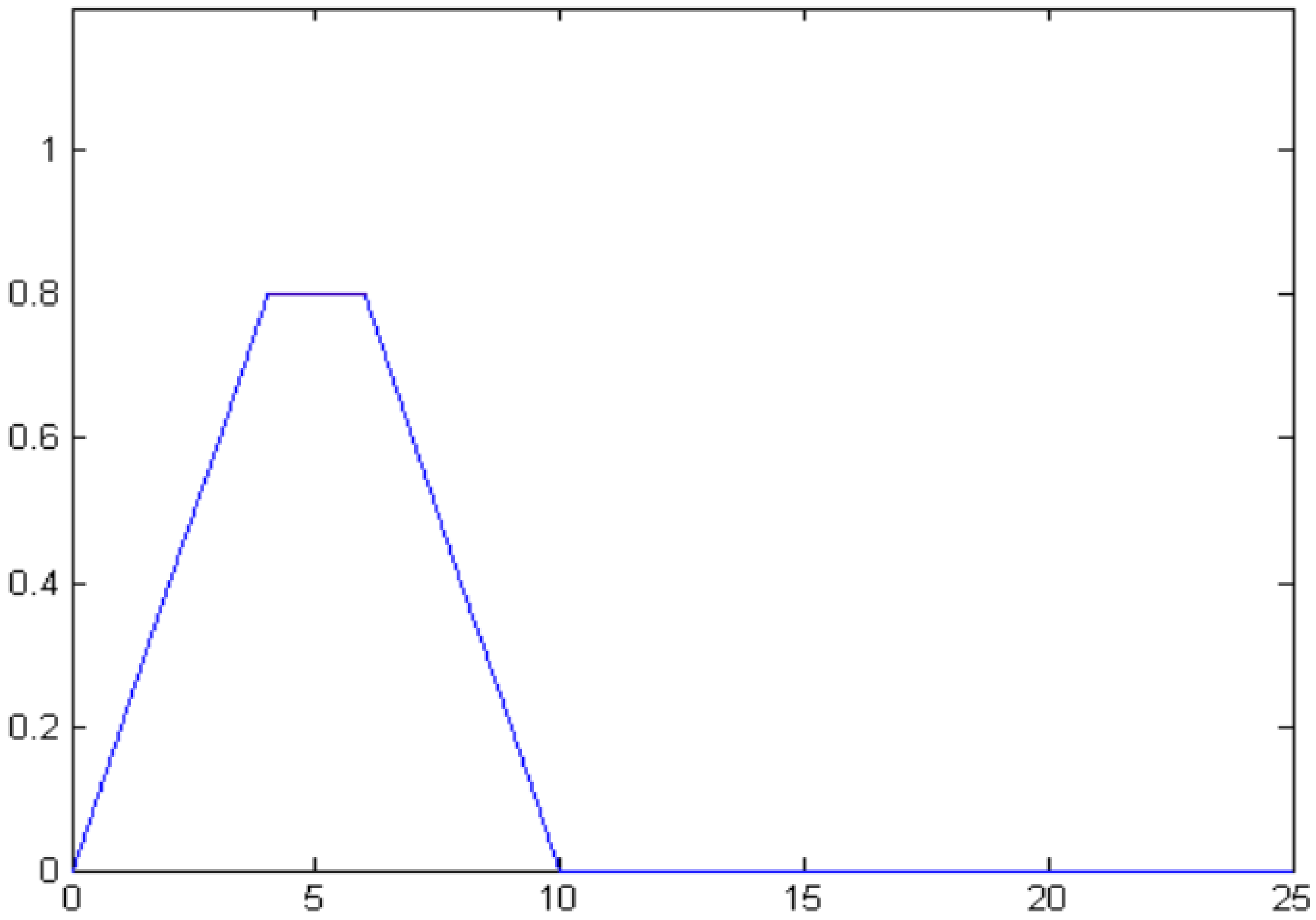
- Apply implication method
- Builds an inferred fuzzy set
- Find the min value for each output MF
- Cut output MF at this value

Min Inference

- Cut at 0.8



Min Inference

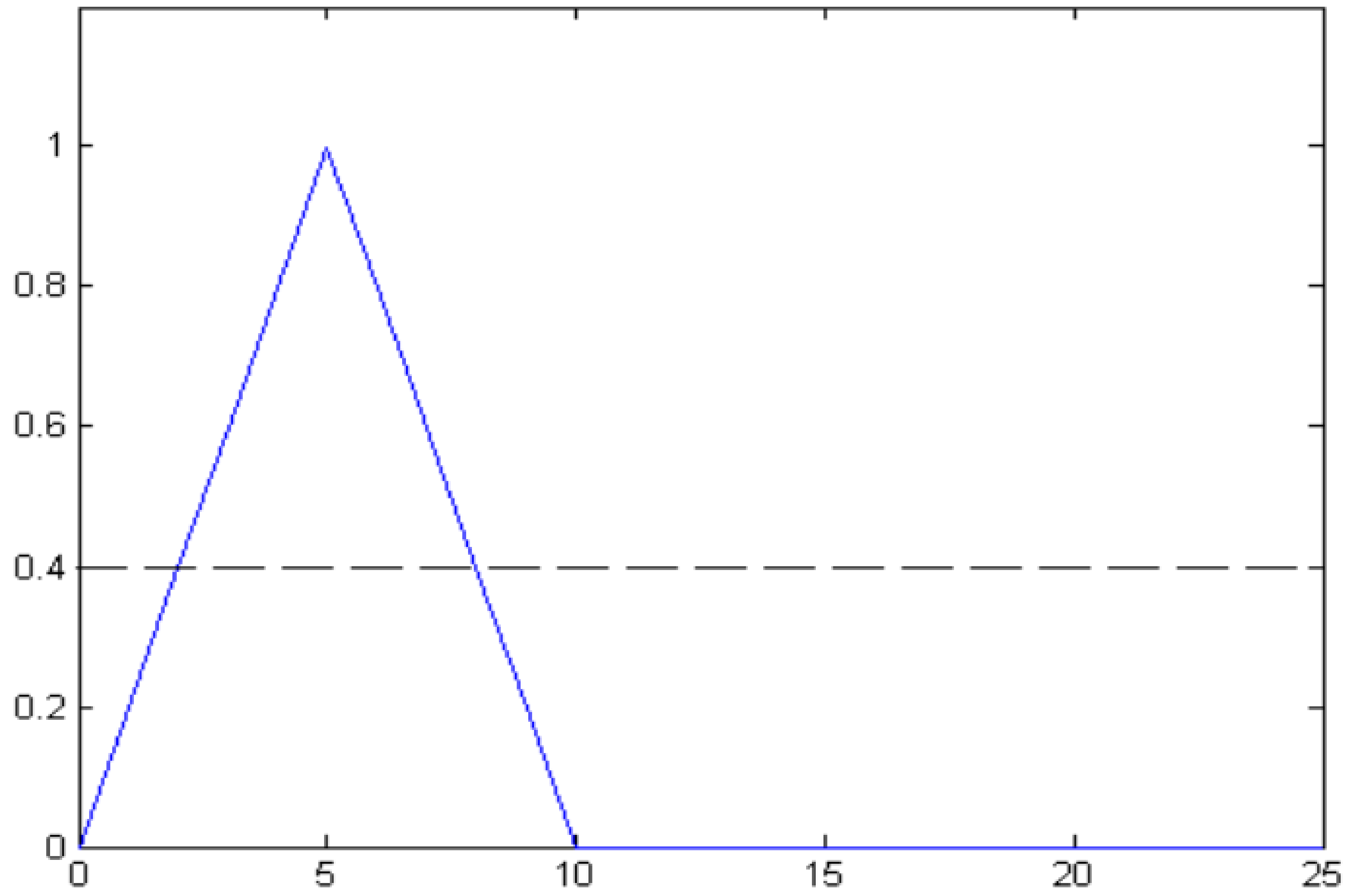


Min Inference

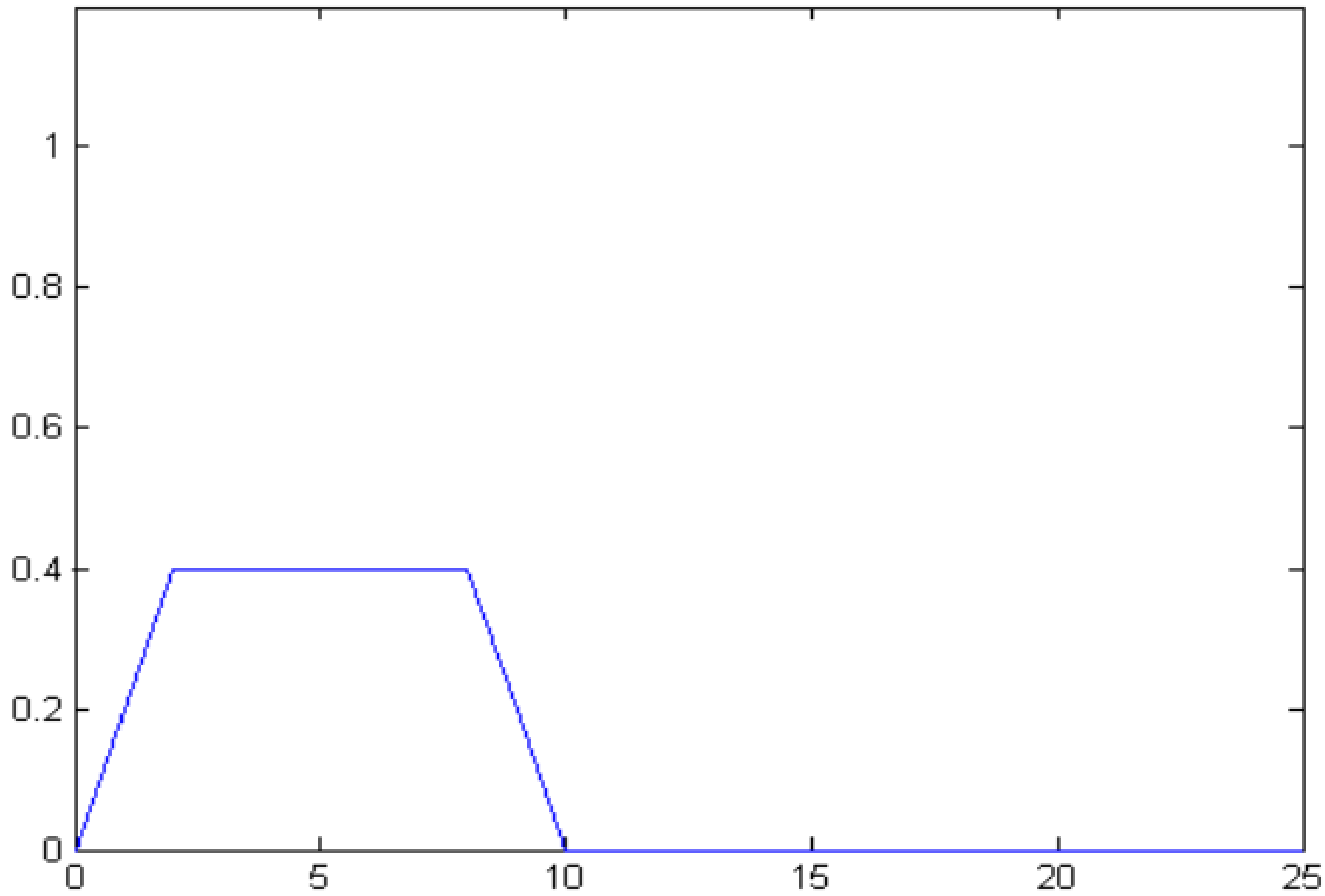
- Corresponding fuzzy set
  - MF = {(0,0),(1,0.2),(2,0.4),(3,0.6),(4,0.8),(5,0.8),(6,0.8),(7,0.6),(8,0.4),(9,0.2),(10,0), (25,0)}

Min Inference

- Degree of support of 0.4



Min Inference



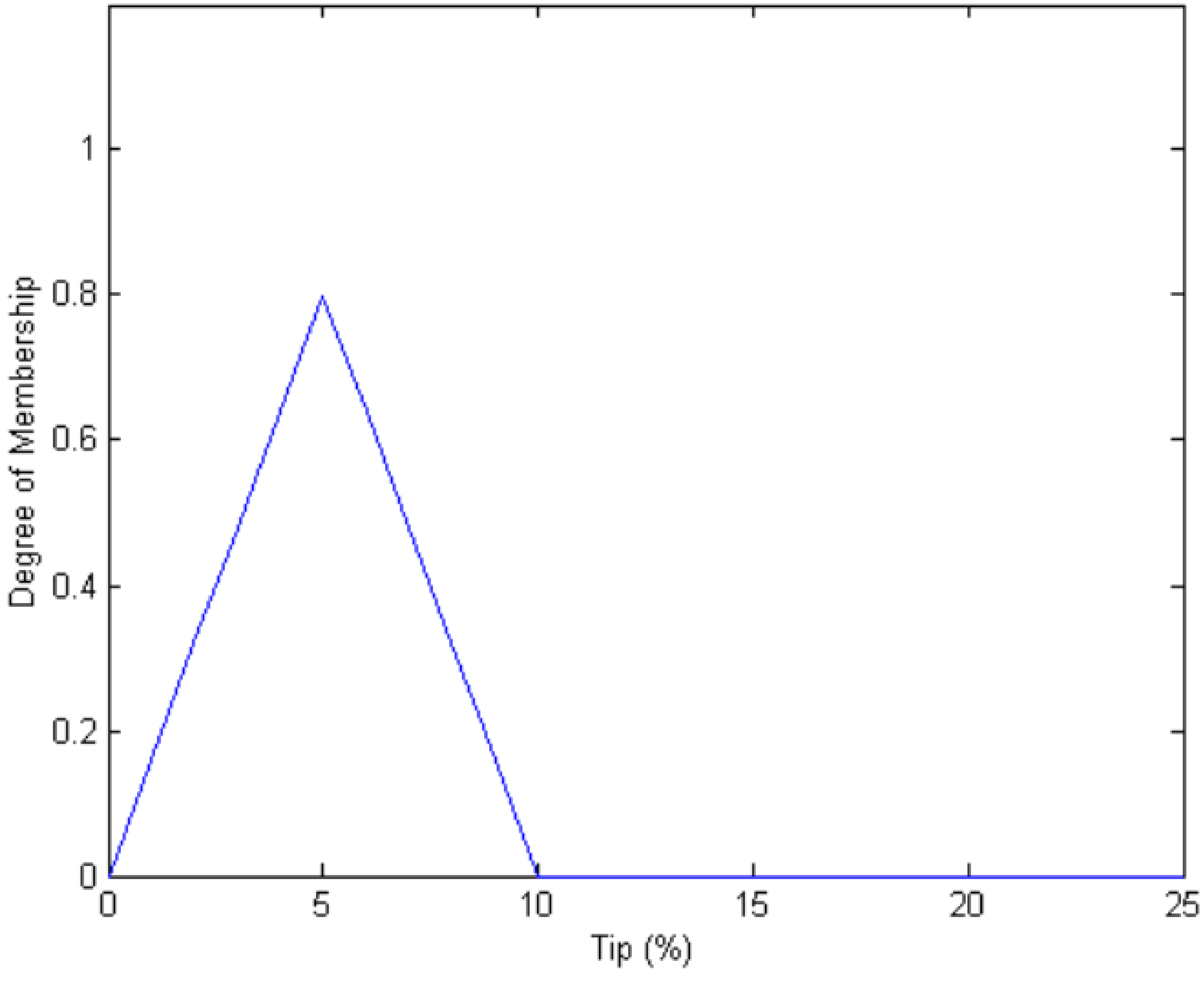
Min Inference

- Corresponding set
    - MF = {(0,0),(1,0.2),(2,0.4),(3,0.4),(4,0.4),(5,0.4),(6,0.4),(7,0.4),(8,0.4),(9,0.2),(10,0), (25,0)}

Fuzzy Inference

- How are things different if we use product inferencing?

Product Inference

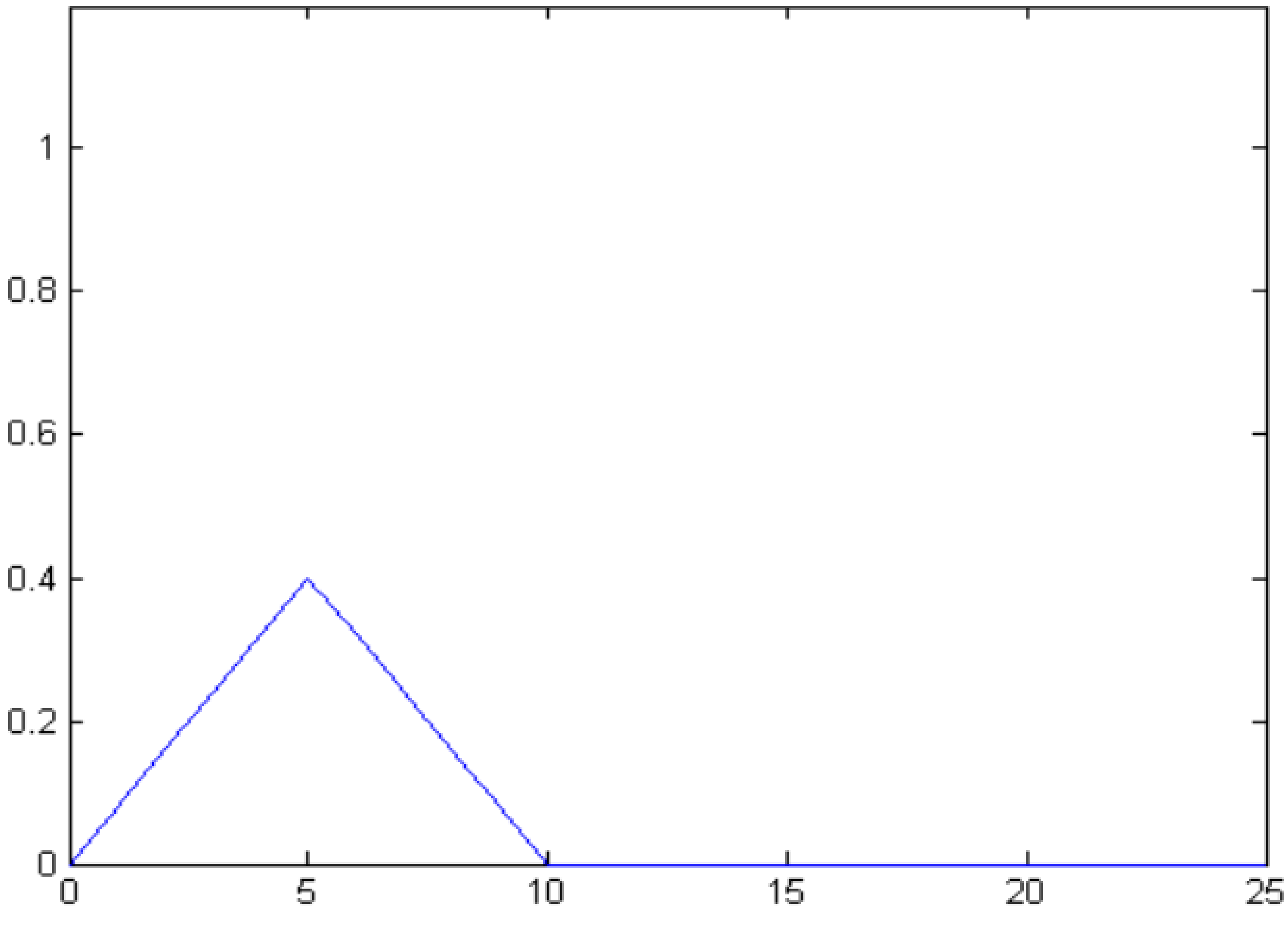


Product Inference

- Corresponding set
  - MF = {(0,0),(1,0.16),(2,0.32),(3,0.48),(4,0.64),(5,0.8),(6,0.64),(7,0.48),(8,0.16),(9,0.16),(10,0),(25,0)}

Product Inference

- Degree of support of 0.4



Product Inference

- Corresponding set
  - MF = {(0,0),(1,0.08),(2,0.16),(3,0.24),(4,0.32),(5,0.4),(6,0.32),(7,0.24),(8,0.16),(9,0.08),(10,0),(25,0)}

Fuzzy Composition

- Aggregates the inferred MF into one
- Two methods of doing this
  - Max
  - Sum

Fuzzy Composition

- MAX takes the max fuzzy value for each value of *v*
  - equivalent to taking the fuzzy values for the highest activated rule for each output fuzzy set
- SUM sums all fuzzy values for each value of *v*
  - can lead to truth values > 1
  - may need to be normalised to [0,1]
    - implications for defuzzification

Fuzzy Composition

- Assume
  - 3 MF attached to the output
    - A, B and C
  - Each MF has been asserted by two different rules
    - 6 rules activated (degrees of support) > 0
  - Degrees of support
    - 0.8, 0.4, 0.6, 0.5, 0.7, 0.3
  - Prod inference used

Fuzzy Composition

- For Set A

v	0	1	2	3	4	5	6	7	8	9	10
$\mu(v)'$	0	0.16	0.32	0.48	0.64	0.8	0.64	0.48	0.32	0.16	0
	0	0.08	0.16	0.24	0.32	0.4	0.32	0.24	0.16	0.08	0

- For Set B

v	7	8	9	10	11	12	13	14	15	16	17
$\mu(v)'$	0	0.12	0.24	0.36	0.48	0.6	0.48	0.36	0.24	0.12	0
	0	0.1	0.2	0.3	0.4	0.5	0.4	0.3	0.2	0.1	0

- For Set C

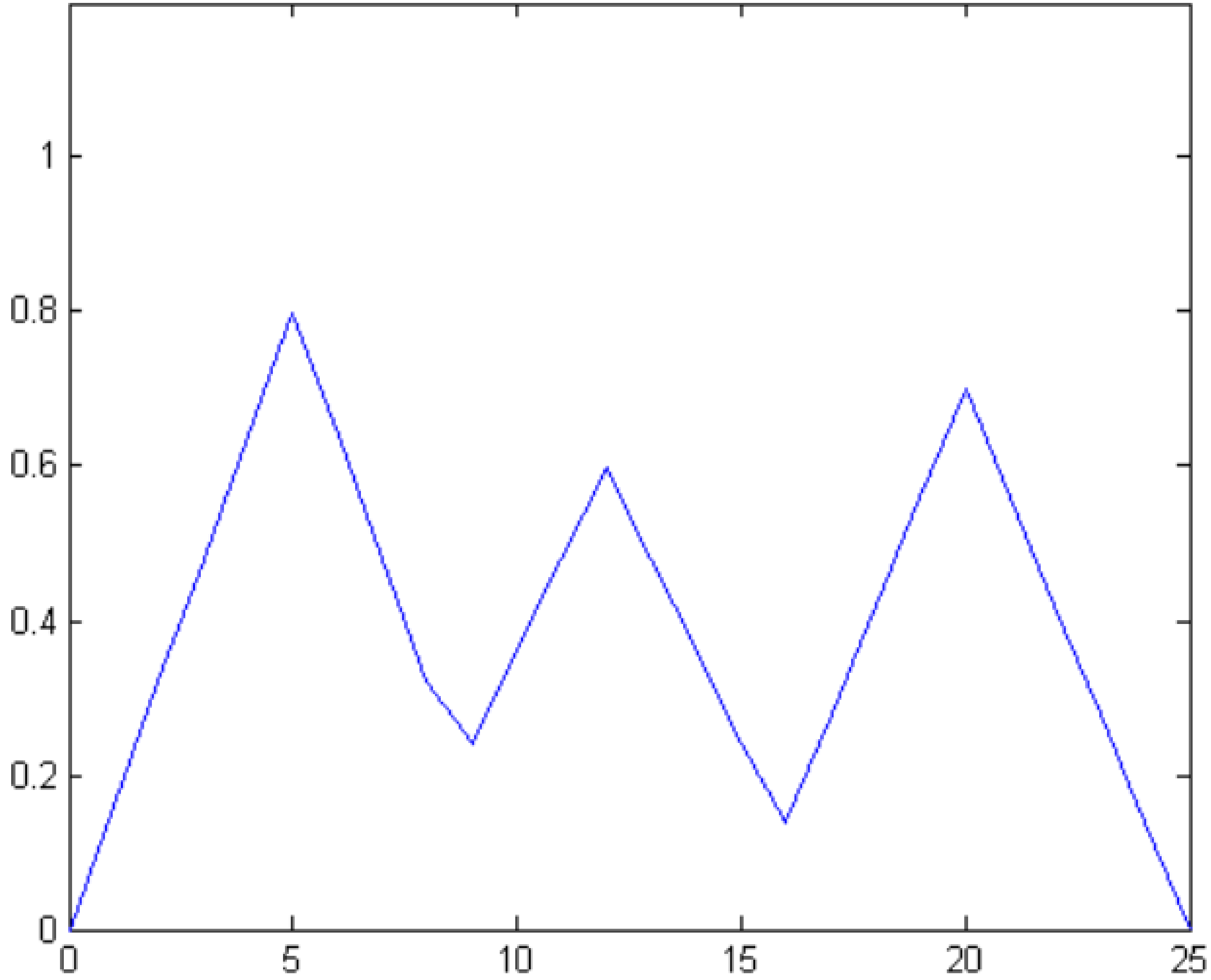
v	15	16	17	18	19	20	21	22	23	24	25
$\mu(v)'$	0	0.14	0.28	0.42	0.56	0.7	0.56	0.42	0.28	0.14	0
	0	0.06	0.12	0.18	0.24	0.3	0.24	0.18	0.12	0.06	0

Max Composition

- MAX composition
  - Take the max of each column

v	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(v)'$	0	0.16	0.32	0.48	0.64	0.8	0.64	0.48	0.32	0.24	0.36	0.48	0.6
v	13	14	15	16	17	18	19	20	21	22	23	24	25
$\mu(v)'$	0.48	0.36	0.24	0.14	0.28	0.42	0.56	0.7	0.56	0.42	0.28	0.14	0

Max Composition

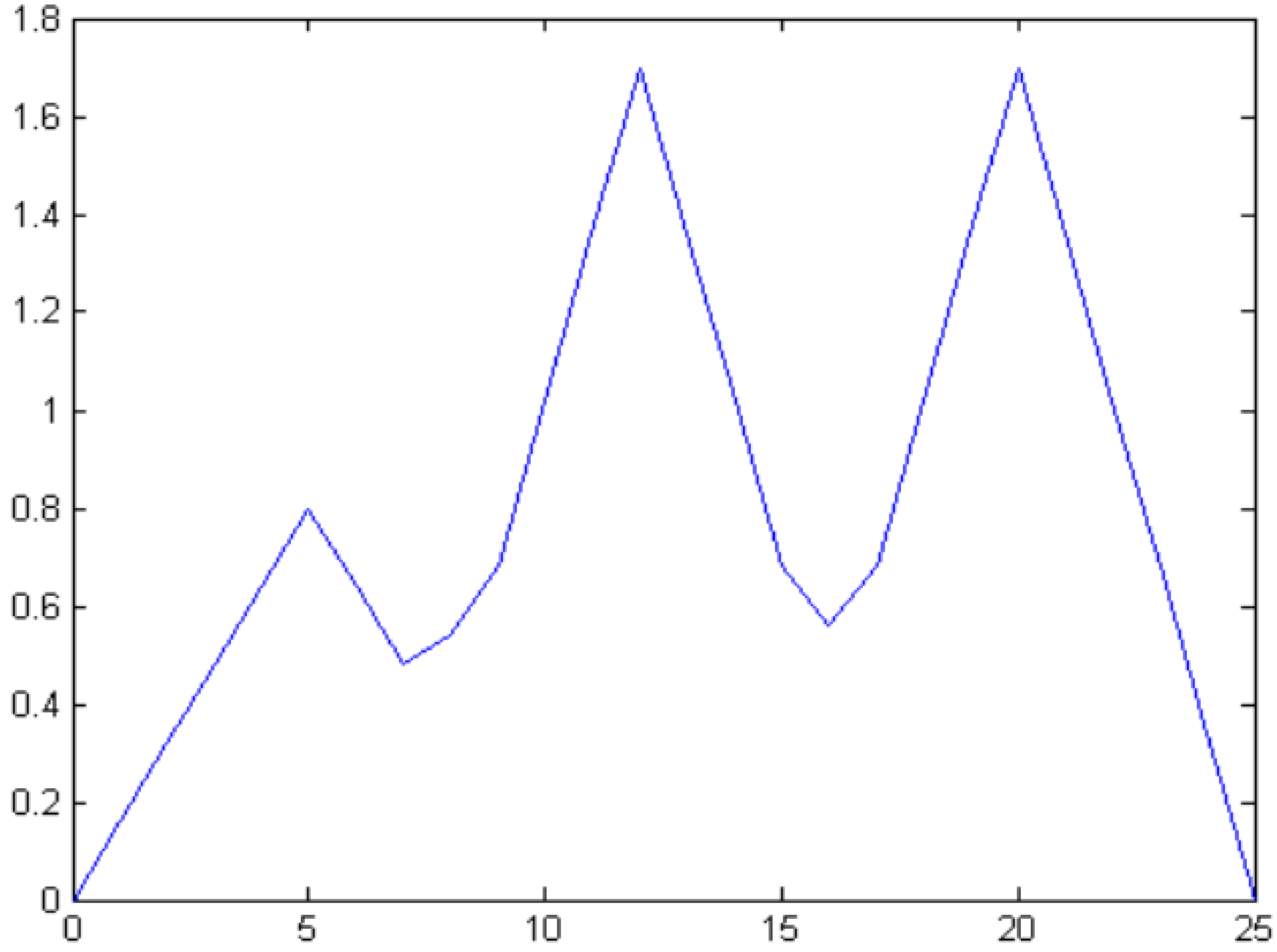


Sum Composition

- Sum composition
  - sum each column

v	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(v)'$	0	0.16	0.32	0.48	0.64	0.8	0.64	0.48	0.54	0.68	1.02	1.36	1.7
v	13	14	15	16	17	18	19	20	21	22	23	24	25
$\mu(v)'$	1.36	1.02	0.68	0.56	0.68	1.02	1.36	1.7	1.36	1.02	0.68	0.34	0

Sum Composition



Defuzzification

- Converts inferred MF into crisp numbers
- Many different types in existence
- Two common ones
  - Centre of Gravity
  - Mean of Maxima

COG Defuzzification

- Centre of Gravity
  - CoG

$$y = \frac{\sum_i^K \mu(v_i) v_i}{\sum_i^K \mu(v_i)}$$

- Where:
  - *y* is the crisp value
  - *K* is the number of items in the fuzzy set

COG Defuzzification

- Applying this to the first composite set

v	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(v)'$	0	0.16	0.32	0.48	0.64	0.8	0.64	0.48	0.32	0.24	0.36	0.48	0.6
$v\mu(v)'$	0	0.16	0.64	1.44	2.56	4	3.84	3.36	2.56	2.16	3.6	5.28	7.2
v	13	14	15	16	17	18	19	20	21	22	23	24	25
$\mu(v)'$	0.48	0.36	0.24	0.14	0.28	0.42	0.56	0.7	0.56	0.42	0.28	0.14	0
$v\mu(v)'$	6.24	5.04	3.6	2.24	4.76	7.56	10.64	14	11.76	9.24	6.44	3.36	0

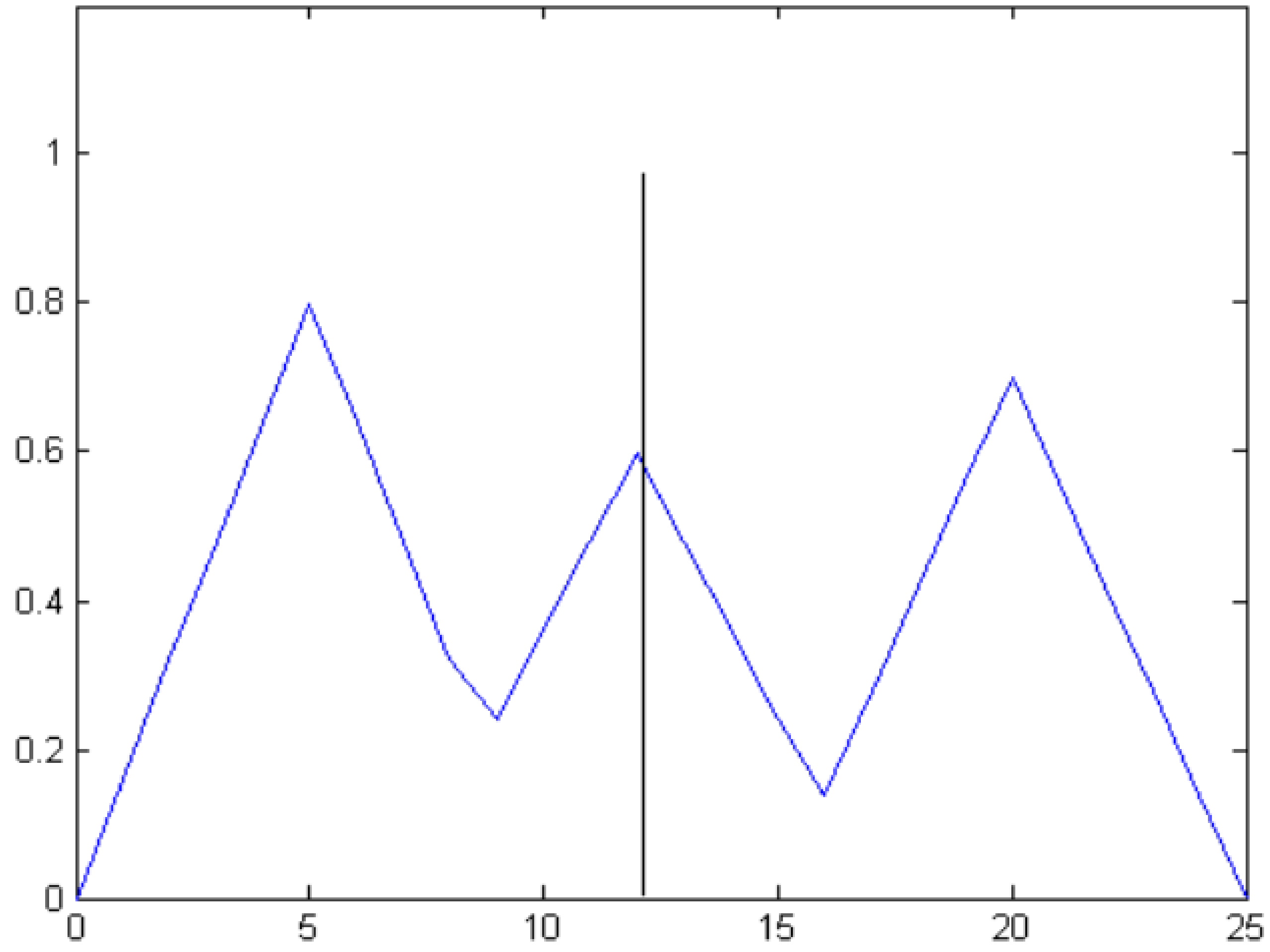
COG Defuzzification

$$\sum_{i}^{K} \mu(v_i) v_i = 121.68$$

$$\sum_{i}^{K} \mu(v_i) = 10.1$$

$$\frac{121.68}{10.1} = 12.05$$

COG Defuzzification



Defuzzification

- Mean of Maxima
  - MoM
- Finds the mean of the crisp values that correspond to the maximum fuzzy values
- If there is one maximum fuzzy value, the corresponding crisp value will be taken from the fuzzy set

MoM Defuzzification

- Applying this to the first composite set
- Maximum fuzzy value is 0.8
- Corresponding crisp value is 4
- This is the value returned by MoM

MoM Defuzzification

- What about sets with > 1 maximum?
- Apply this to the third composite set

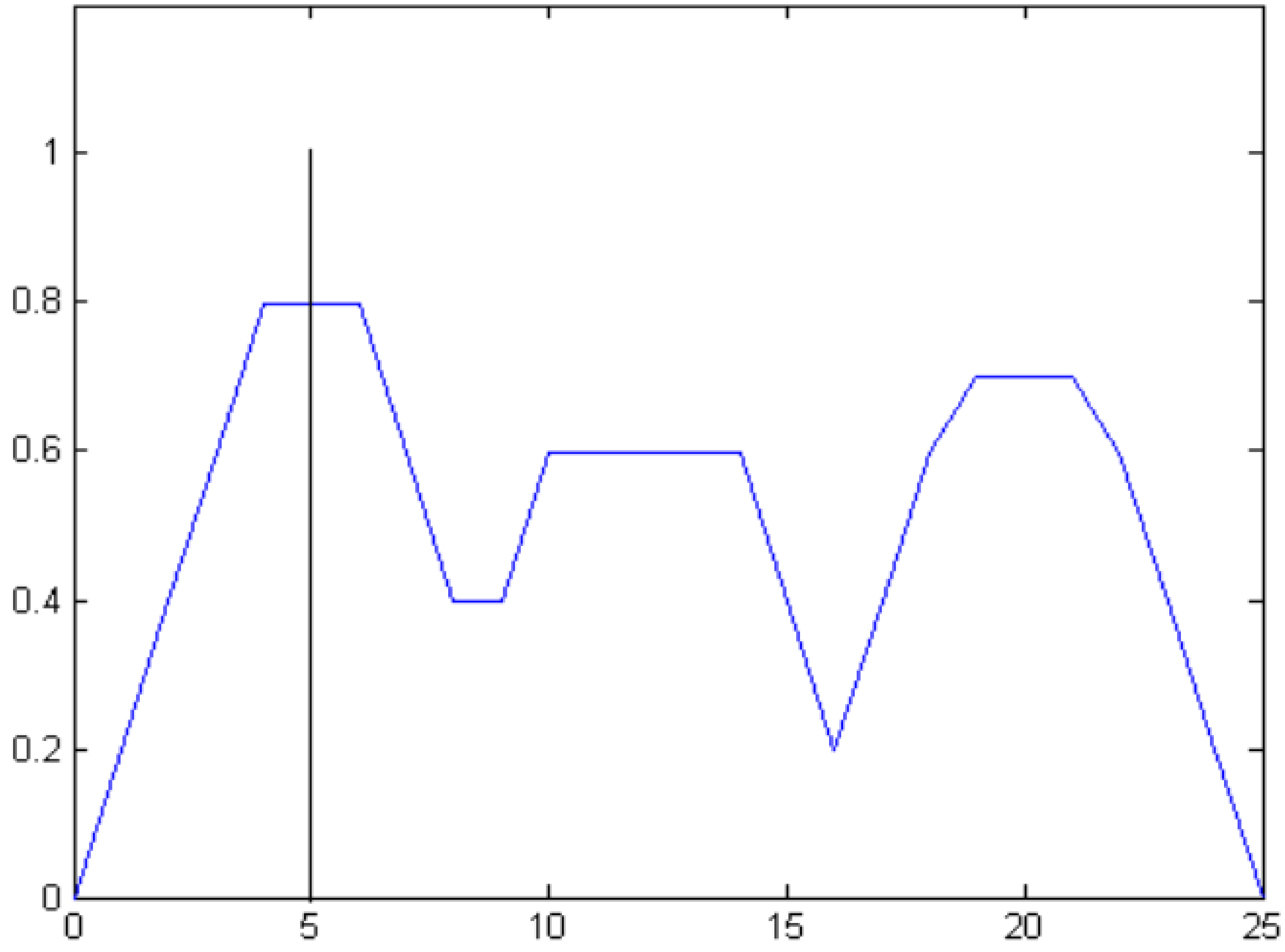
v	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(v)'$	0	0.2	0.4	0.6	0.8	0.8	0.8	0.6	0.4	0.4	0.6	0.6	0.6
v	13	14	15	16	17	18	19	20	21	22	23	24	25
$\mu(v)'$	0.6	0.6	0.4	0.2	0.4	0.6	0.7	0.7	0.7	0.6	0.4	0.2	0

MoM Defuzzification

- Maximum fuzzy value if 0.8
- Corresponding crisp values are
  - 4, 5 and 6

$$y = \frac{4+5+6}{3} = 5$$

MoM Defuzzification



Summary

- Fuzzy rules match fuzzy antecedents to fuzzy consequents
- Degree to which antecedents are true determine the degree of support
- Fuzzy logic functions are used to determine this

Summary

- Fuzzy inference involves calculating an output fuzzy set
- Different inference process produces different inferred MF
- Two inferences processes are
  - max-min
  - Max-prod

Summary

- Two common composition methods
  - MAX
  - SUM
- Inference methods described by combining inference & composition methods
  - max-min (or min-max)
  - max-prod
- Defuzzification converts a composed MF to a single crisp value

Summary

- Different defuzzification methods produce different crisp values
  - sometimes wildly different
- Two different defuzzification methods
  - Centre of Gravity
    - CoG
  - Mean of Maxima
    - MoM